Int. J. Solids Structures Vol. 27, No. 15, pp. 1925–1941, 1991
Printed in Great Britain.

0020-7683/91 $3.00 + .00

DELAMINATION CRACK ORIGINATING FROM TRANSVERSE CRACKING IN CROSS-PLY COMPOSITE LAMINATES UNDER EXTENSION

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(*Received* 1 *February* 1990; *in revised form* 22 *August* 1990)

Abstract—Based upon the Stroh formalism for anisotropic elastic materials and upon the method of eigenfunction expansion, the stress redistribution due to delamination cracks originating from transverse cracking is examined from $[90/0]_s$ and $[0/90]_s$ laminates under extension. The structure of the solution, in the form of a series expansion, is determined from the eigenvalue equation resulting from appropriate near-field conditions. To complete the solution, use is made of a boundary collocation technique in conjunction with the eigenfunction series that includes a large number of terms, enough to represent the elastic state throughout the appropriate domain concerned. The fracture mechanics parameters, such as stress intensity factors and energy release rates, are calculated and the major characteristics of stress distribution are discussed. The stability of delamination cracks is examined for varying ratios of ply thickness in terms of the energy release rate.

1. INTRODUCTION

Fracture analysis of an anisotropic composite laminate has gained substantial attention in association with the understanding of its fundamental fracture behavior, which is of critical importance for the design of composite structures. In particular, transverse cracking and delamination fracture have been among the subjects under intensive investigation during the last two decades. For example, stress singularities were first reported by Ting and Hoang (1984) for transverse cracks, and Wang and Choi (1983a) treated delamination cracks in anisotropic composite laminates. Various problems concerning transverse cracking and delamination damage, including crack initiation and growth, stiffness reduction and other changes of structural properties, have been discussed by many researchers; the number of works along this line is too great to give individual citation herein. Many important contributions may be found in the thesis by Lim (1988), for example.

Recently, the asymptotic stress field near transverse cracks occurring in the 90° ply of cross-ply laminates was examined by Im (1989) and Im and Kim (1989). As the load increases, the transverse cracks in the 90° ply of a cross-ply laminate that are arrested at the interface tend to kink into the delamination cracks along the ply interface (Lim, 1988). The purpose of this work is to obtain the stress redistribution under extension in the presence of such interfacial cracks originating from transverse cracking, and to examine the fracture behavior, including the stability of delamination cracks and the influence of geometric parameters. It is assumed that these interface cracks run through the width of a laminate, with a uniform spacing and crack length. The asymptotic structure of the solution is obtained, under the assumption of plane strain elasticity, from the Stroh formalism of anisotropic elasticity and the eigenfunction expansion [see Ting (1986) or references therein]; appropriate near-field conditions are imposed to lead to the eigenvalue equations, which determine the structure of the asymptotic solution, including the stress singularity. To complete the solution, use is then made of a boundary collocation technique in conjunction with the eigenfunction series that includes a large number of terms, enough to cover the elastic state of the far field as well as of the near field. The convergence behavior of the solution, including the influence of the number of eigenvalues truncated in the series solution and the number of collocation stations, is demonstrated through numerical examples and the role of the two translational planar rigid body modes, in association with the solution accuracy, is critically examined. The characteristics of the singular stress field near the crack tip are briefly discussed for $[90/0]_s$ and $[0/90]_s$, respectively; under the given loading of extension, the delamination crack turns out to be opened for $[90/0]_s$ laminates, while $[0/90]_s$ laminates are found to have a fully closed delamination crack. For the opened delamination crack in $[90/0]_s$ laminates which has an oscillatory singularity, the stress intensity for an interfacial crack using the analogy of linear elastic fracture mechanics, as proposed by Rice (1988), is computed. The effect of ply thickness is assessed in terms of the energy release rate, and the stability of delamination cracks is examined under a fixed load condition.

2. STATEMENT OF THE PROBLEM AND BASIC EQUATIONS

Consider the two types of cross-ply composite laminate, $[90/0]_s$ and $[0/90]_s$, subjected to extension. As the load increases, there will occur numerous transverse cracks running parallel to the fiber orientation of the 90° ply, with an approximately uniform spacing along the length of the laminates. These cracks, terminating perpendicular to the ply interface because of the stronger 0° ply, tend to develop into delamination cracks along the interface as the load increases further. Figure 1 is typical of such delamination cracks originating from transverse cracking (Lim, 1988). It is noticed that the cracks run through the width of a laminate. We will examine the stress redistribution in the laminate due to the presence of such interfacial cracks under extensional loading. To simplify the problem, we assume that the cracks are uniformly arranged with the configuration symmetric about the mid-plane, as shown in Fig. 2, so that the overall arrangement is obtained by repetition of the representative unit cell.

We take a rectangular Cartesian coordinate system with the origin at one of the crack tips. We choose the x_1 axis to be along the length of the composite laminates or along the loading direction, while the x_2 axis is taken to be along the direction of the laminate thickness, and then the x_3 axis is along the laminate width, which is parallel to the cracks (see Fig. 2). Each ply of the composite laminates lies in a plane parallel to the x_1–x_3 plane, and the ply orientation θ is defined to be the counter-clockwise angle, viewed from above, that the fiber direction makes with the x_1 axis. We assume that the laminate dimension in the x_3 direction (laminate width) is sufficiently large compared with the laminate thickness, and that the laminate is assumed to be in the state of plane strain on the x_1–x_2 plane.

The plane problems of a composite body comprising isotropic materials, including stress distribution around composite wedges, were treated by many researchers, e.g. Zak and Williams (1963), Bogy (1971), Cook and Erdogan (1972) and Dempsey and Sinclair (1979). For an anisotropic composite body, the out-of-plane displacement is in general nonzero due to the associated coupling in the stress–strain relation. This is so even when the normal strain in the out-of-plane direction is assumed to be zero (which is the case for deformations that depend upon only the two coordinates on the plane). Among the approaches in analyzing such elastic deformations, Lekhnitskii's complex potentials were used by Wang and Choi and their associates [e.g. Wang and Choi (1982), Wang and Yuan (1983)] to examine the boundary-layer effect in the free-edge problem. Ting and his associates, on the other hand, used Stroh's approach (Stroh, 1962) to treat the stress singularities near various anisotropic composite wedges and cracks [Ting (1986) and references cited therein]. This approach has the advantage, as shown by Ting (1986), that it leads to a simple and neat formulation of problems in anisotropic elasticity. Due to such an advantage, we use the Stroh formalism to determine the structure of the solution for the present problem.

Let u_i, ε_{ij} and σ_{ij} denote the Cartesian components of displacement, strain and stress, respectively. For deformations that depend upon two coordinates x_1 and x_2 only, we have the governing equations:

$$\text{Equilibrium equation, } \sigma_{i1,1}+\sigma_{i2,2} = 0 \text{ (no body force)}, \tag{1a}$$

$$\text{Strain–displacement relation, } \varepsilon_{ij} = (u_{i,j}+u_{j,i})/2, \tag{1b}$$

$$\text{Stress–strain relation, } \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad C_{ijkl} = C_{jikl} = C_{klij}, \tag{1c}$$

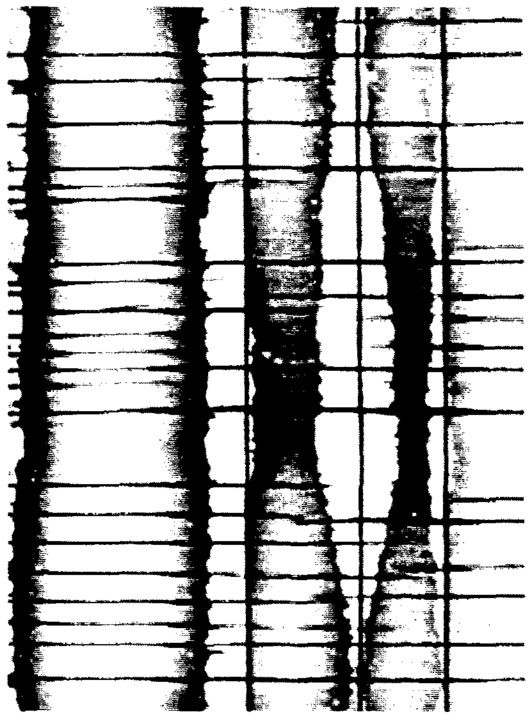

Fig. 1. Typical delamination cracks originating from transverse cracking (Lim, 1988) (white areas represent the delamination zone and solid lines the transverse cracks).

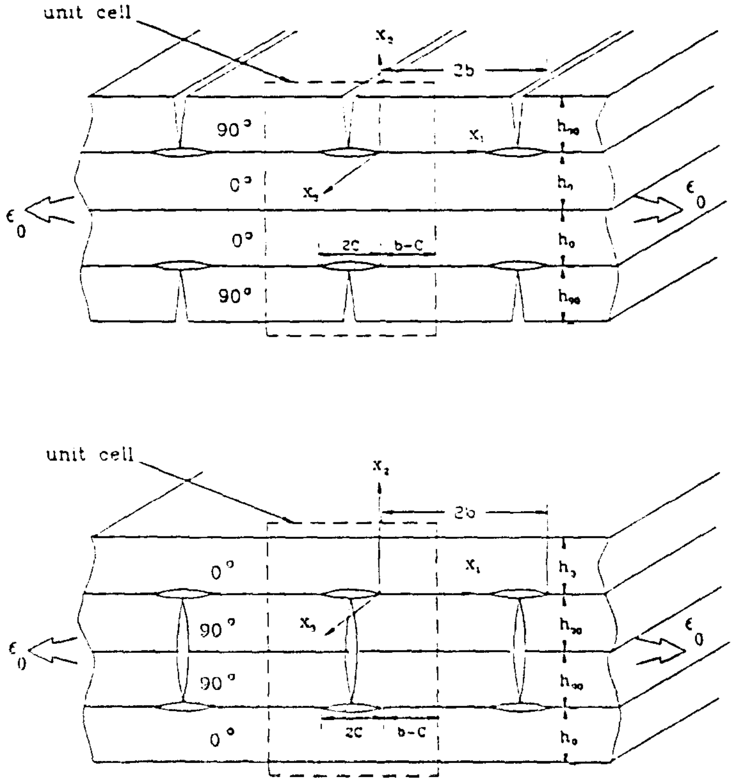

Fig. 2. Delamination cracks originating from transverse cracking in $[90/0]_s$ and $[0/90]_s$ ($\varepsilon_0 = \bar{u}/b$).

where C_{ijkl} are the fourth-order stiffness tensors, and the comma indicates partial differentiation with respect to x_i. Introducing the "collapsed" representation, we may write the stress–strain relation as

$$\sigma_i = C_{ij}\varepsilon_j, \quad \varepsilon_i = S_{ij}\sigma_j, \tag{2a,b}$$

where

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}, \tag{2c,d}$$

and C_{ij} and S_{ij} are the stiffness and the compliance matrix. For orthotropic materials like $[90/0]_s$ and $[0/90]_s$ laminates, we have

$$C_{14} = C_{15} = C_{16} = C_{24} = C_{25} = C_{26} = C_{34} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0. \tag{3}$$

For such materials, the displacement component u_3 is decoupled from u_1, u_2, and the plane strain assumption can be made for extension along the x_1 axis, as in the present deformation. Following Ting and Chou (1981), we can show that the general solution takes the form for the present problem,

$$u_\alpha = \sum_{k=1}^{4} v_{\alpha k} f_k(z_k), \quad z_k = x_1 + \mu_k x_2 \quad (k = 1, 2, 3, 4), \tag{4a,b}$$

$$\sigma_{\alpha\beta} = \sum_{k=1}^{4} \tau_{\alpha\beta k} \frac{\mathrm{d}f(z_k)}{\mathrm{d}z_k}, \quad \tau_{\alpha\beta k} = (C_{\alpha\beta\gamma 1} + \mu_k C_{\alpha\beta\gamma 2}) v_{\gamma k} \quad (\text{no sum on } k), \tag{5a,b}$$

where the Greek letters α, β, γ take the value 1 or 2, and summation is implied on repeated Greek indices only; μ_k are the eigenvalues to be determined, and $f_k(z_k)$ is a function of z_k. Substituting eqns (5a,b) into eqn (1a), we obtain the equations:

$$D_{\alpha\beta}(\mu_k) v_{\beta k} = 0,$$
$$D_{\alpha\beta}(\mu_k) = C_{\alpha 1\beta 1} + \mu_k (C_{\alpha 1\beta 2} + C_{\alpha 2\beta 1}) + \mu_k^2 C_{\alpha 2\beta 2}. \tag{6a,b}$$

For the existence of nontrivial solutions, we have

$$\det [D_{\alpha\beta}(\mu_k)] = |D_{\alpha\beta}(\mu_k)| = 0, \tag{7}$$

and the solutions of this quartic equation yield the two pairs of complex conjugate eigenvalues (Stroh, 1962),

$$\bar{\mu}_{\alpha+2} = \mu_\alpha \quad (\alpha = 1, 2). \tag{8}$$

The associated eigenvector $v_{\beta k}$ can now be obtained from (6a) through a proper normalization. Using the collapsed symmetry relation (3), we obtain the explicit expression for eqn (7)

$$\mu_k^4 + \mu_k^2 \{C_{11}/C_{66} - C_{12}^2/(C_{66}C_{22}) - 2C_{12}/C_{22}\} + C_{11}/C_{22} = 0. \tag{9}$$

The four eigenvalues μ_k appear as two pairs of complex conjugates, and they are purely imaginary if the inequality

$$\{C_{11}/C_{66} - C_{12}^2/(C_{66}C_{22}) - 2C_{12}/C_{22}\}^2 - 4(C_{11}/C_{22}) \geqslant 0$$

holds among the stiffness components C_{ij}; otherwise they have a nonzero real part. For the advanced fibrous composite materials such as graphite epoxy and boron epoxy, the above inequality holds and therefore the values of μ_k are assumed to be purely imaginary hereafter. It is worthwhile to note that these four eigenvalues μ_k are the constants that make eqn (4b) the complex characteristics of the governing elliptic partial differential equation for Lekhnitskii's complex stress potential (Lekhnitskii, 1963).

3. ASYMPTOTIC EIGENFUNCTION EXPANSION AND BOUNDARY COLLOCATION SOLUTION

In this section, the asymptotic eigenfunction expansion for the function $f_k(z_k)$ is examined near the delamination crack tip, and use is made of this expansion to obtain the complete numerical solution with the aid of the boundary collocation method. We assume a power-type eigenfunction for $f_k(z_k)$ as given by (Wang and Choi, 1982; Ting, 1986)

$$f_k(z_k) = \sum_{n=1}^{I} C_{kn} z_k^{\delta_n + 1}/(\delta_n + 1), \tag{10}$$

which leads to the expression for the displacement and stress field

$$u_\alpha^{(m)} = \sum_{n=1}^{\infty}\sum_{k=1}^{4} C_{kn}^{(m)} v_{\alpha k}^{(m)} z_k^{(m)\delta_n+1}/(\delta_n+1) \quad (m = 1,2), \tag{11a}$$

$$\sigma_{\alpha\beta}^{(m)} = \sum_{n=1}^{\infty}\sum_{k=1}^{4} C_{kn}^{(m)} \tau_{\alpha\beta k}^{(m)} z_k^{(m)\delta_n} \quad (m = 1,2), \tag{11b}$$

where the superscript $m = 1, 2$ indicates the upper and the lower ply, respectively. Here the eigenvalues δ_n are to be determined from the so-called "near-field" conditions around the crack tip, including the traction or displacement conditions on the crack surface and on the ply interface. The coefficient C_{kn} is dependent upon the associated eigenvalues δ_n, and can be determined within an arbitrary constant when δ_n is obtained.

We assume that there is no friction on the delamination crack surface and that the two plies are rigidly bonded along the rest of the ply interface. Then the near-field conditions for the present case may be written as

$$\sigma_{22}^{(1)}(x_1, 0^+) = \sigma_{22}^{(2)}(x_1, 0^-) = 0, \quad \text{or} \quad [\sigma_{22}(x_1, 0)] = [u_2(x_1, 0)] = 0$$
$$\text{(on the crack surface, } x_1 \leqslant 0), \tag{12}$$

$$\sigma_{12}^{(1)}(x_1, 0^+) = \sigma_{12}^{(2)}(x_1, 0^-) = 0, \quad \text{(on the crack surface, } x_1 \leqslant 0), \tag{13}$$

$$[\sigma_{22}(x_1, 0)] = [\sigma_{12}(x_1, 0)] = 0, \quad [u_1(x_1, 0)] = [u_2(x_1, 0)] = 0$$
$$\text{(on the interface, } x_1 \geqslant 0), \tag{14a,b}$$

where the bracket [•] denotes the discontinuity of the quantity in it across the ply interface, e.g. $[\sigma_{22}(x_1, 0)] = \sigma_{22}^{(1)}(x_1, 0^+) - \sigma_{22}^{(2)}(x_1, 0^-)$. The first condition in eqn (12) is valid for the opened region of the crack surface and the latter is for the closed region. Substituting the expressions for displacement and stress (11) into the above near-field conditions, we obtain a system of 8×8 homogeneous linear equations:

$$\Delta_{ij}(\delta_n) B_{jn} = 0 \quad (i, j = 1\text{–}8), \quad B_{kn} = C_{kn}^{(1)}, \quad B_{(k+4)n} = C_{kn}^{(2)} \quad (k = 1\text{–}4). \tag{15a}$$

For the existence of nontrivial solutions, we have

$$|\Delta_{ij}(\delta_n)| = 0, \tag{15b}$$

which determines the eigenvalues δ_n. When the eigenvalues δ_n are known, within unknown constants the eigenvectors C_{jn} are computed from eqn (15a) and the asymptotic form of the solutions for the stress and displacement is given by eqns (11a,b). From the structure of $\Delta_{ij}(\delta_n)$, we can show that if δ_n is a root of the characteristic equation, so is its complex conjugate $\bar{\delta}_n$, and the expressions (11a,b) for the stress and displacement become real. To take only the real part of eqns (11a,b) for convenience, we may introduce

$$C_{kn}^{(\alpha)} = \tfrac{1}{2}(\gamma_{1n} - i\gamma_{2n}) b_{kn}^{(\alpha)} \quad \text{for complex } \delta_n, \quad \operatorname{Im}[\delta_n] > 0$$
$$C_{kn}^{(\alpha)} = \tfrac{1}{2}\gamma_{3n} b_{kn}^{(\alpha)} \quad \text{for real } \delta_n \tag{16}$$

where $b_{kn}^{(\alpha)}$ is the solution of eqn (15a), computed by a proper normalization, and γ_{1n}, γ_{2n} and γ_{3n} are constants to be determined to complete the solution. The expressions for the stress and displacement (11a,b) are then written as

$$u_\alpha^{(m)} = \sum_{n=1}^{\infty} Q_{n\alpha}^{(m)}, \quad \sigma_{\alpha\beta}^{(m)} = \sum_{n=1}^{\infty} P_{n\alpha\beta}^{(m)} \quad (m = 1, 2), \tag{17a,b}$$

where $Q_{n\alpha}^{(m)}$ and $P_{n\alpha\beta}^{(m)}$ are given by

$$Q_{n\alpha}^{(m)} = \gamma_{1n}\,\mathrm{Re}\,[\psi_{n\alpha}]+\gamma_{2n}\,\mathrm{Im}\,[\psi_{n\alpha}],\quad P_{n\alpha\beta}^{(m)} = \gamma_{1n}\,\mathrm{Re}\,[\phi_{n\alpha\beta}]+\gamma_{2n}\,\mathrm{Im}\,[\phi_{n\alpha\beta}],$$

$$\psi_{n\alpha} = \sum_{k=1}^{2} [b_{kn}^{(m)} v_{\alpha k}^{(m)} z_k^{(m)\delta_n+1} + b_{(k+2)n}^{(m)} \bar{v}_{\alpha k}^{(m)} \bar{z}_k^{(m)\delta_n+1}]\,/(\delta_n+1),$$

$$\phi_{n\alpha\beta} = \sum_{k=1}^{2} [b_{kn}^{(m)} \tau_{\alpha\beta k}^{(m)} z_k^{(m)\delta_n} + b_{(k+2)n}^{(m)} \bar{\tau}_{\alpha\beta k}^{(m)} \bar{z}_k^{(m)\delta_n}]$$

if δ_n is complex, or

$$Q_{n\alpha}^{(m)} = \gamma_{3n}\,\mathrm{Re}\left[\sum_{k=1}^{2} b_{kn}^{(m)} v_{\alpha k}^{(m)} z_k^{(m)\delta_n+1}/(\delta_n+1)\right]$$

and

$$P_{n\alpha\beta}^{(m)} = \gamma_{3n}\,\mathrm{Re}\left[\sum_{k=1}^{2} \{b_{kn}^{(m)} \tau_{\alpha\beta k}^{(m)} z_k^{(m)\delta_n}\}\right] \quad \text{if } \delta_n \text{ is real.}$$

The unknown constants γ_{in} can be determined by matching the asymptotic expressions with the far-field conditions (the conditions at the remote boundary) through the boundary collocation technique (Wang and Choi, 1982), or through special finite element methods in which the present asymptotic solutions are embedded in a singular crack tip element (Wang and Yuan, 1983), or in which the asymptotic solutions are introduced over the entire domain (Stolarski and Chiang, 1989). Extensive lists of the literature on the boundary collocation methods and the special finite element techniques may be found in Chung (1988) and Atluri and Nakagaki (1986). We use the boundary collocation technique for the present problem here because of its simplicity. A brief description of this technique is given in Appendix A for self-sufficiency of the present development, and for details the readers may be referred to Wang and Choi (1982).

The power-type expansion (10) fails to be complete when the algebraic multiplicity is greater than the geometric multiplicity, that is, there are not enough sets of power-type eigenvectors associated with these multiple eigenvalues [see Dempsey and Sinclair (1979), Ting and Chou (1981) and the references cited therein]. Dempsey and Sinclair (1979) resolved this difficulty by introducing logarithmic eigenfunctions, which ensure the existence of the sets of eigenvectors, enough to span the solution. Subsequently this was extended to the problem of anisotropic composite laminates by Ting and Chou (1981). The existence of the logarithmic eigenfunctions can be examined by calculating the algebraic multiplicity of the eigenvalues and the rank of the associated coefficient matrices $\Delta_{ij}(\delta_n)$ in (15).

An essential prerequisite for the success of the aforementioned boundary collocation technique is that the asymptotic representation (17a,b) should be complete in the sense that it can represent the elastic field over the entire domain concerned, including the far-field region as well as the near-field region. For example, the present boundary collocation method is difficult to apply if there is another singular region or a strong boundary layer in the far field, which the eigenfunction expansion (17a,b) fails to represent; for the single expression for the elastic state valid throughout the entire domain is not available. For such a case, we may use the hybrid F.E.M. wherein each singular or boundary-layer region is modeled as a special element into which the associated asymptotic solution is incorporated, or the enriched F.E.M. wherein the asymptotic solutions are defined over the entire domain. Roughly speaking, we can approximate the far field by retaining a sufficient number of terms in the near-field eigenfunction series, as long as the asymptotic eigenfunction series is complete in the near field and there are no strong boundary layers other than the near-field singular region. For such cases, we in general have no difficulties in applying the boundary collocation method in conjunction with the asymptotic representation in the near field.

Another important issue related to the completeness is the treatment of rigid body modes. If only traction conditions are involved in the remote boundary, such as in the free-edge problems (Wang and Choi, 1982), we exclude rigid body modes from the eigenfunction representation because they are indeterminate under traction boundary conditions. If there are, however, any displacement boundary conditions involved in the remote boundary, which is the case for the present problem, we have to incorporate all relevant rigid body displacements in the asymptotic representation (11a); otherwise the asymptotic solution for displacement (11a) or (17a) fails to be complete and this will lead to an inaccurate solution. The power-type expansion (11a), however, excludes the two displacement modes corresponding to rigid translation in the x_1 and x_2 directions, although these rigid displacements are both compatible with the near-field conditions (12), (13) and (14a,b). Such incompleteness is due to the inherent limitation of assumption (10) wherein (δ_n+1) is introduced in the denominator and therefore the constant translational rigid displacements are excluded. To rectify this, we modify the asymptotic solution (17a) as

$$u_\alpha^{(m)} = \sum_{n=0}^{\infty} Q_{n\alpha}^{(m)}, \quad Q_{0\alpha}^{(m)} = u_\alpha^0 \tag{18}$$

where u_α^0 $(\alpha = 1, 2)$ is the constant translational rigid displacement or the crack tip displacement in the x_α direction, and $Q_{n\alpha}^{(m)}$ $(n \geqslant 1)$ is the same as before. The significance of including these terms is apparent in the light of completeness, and their quantitative effect will be illustrated through numerical examples.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we determine the unknown free constants γ_{1n}, γ_{2n} and γ_{3n} through boundary collocation to complete the solution. We will confirm the convergence of the boundary collocation solution, and discuss the nature of the near-tip singular fields in terms of the stress intensity, and the fracture behavior of delamination cracks, including the stability of cracks and the effects of ply thickness upon the energy release rates.

For the numerical computation, we use the following material data for the graphite epoxy T300/5208 (Whitcomb, 1987):

$$E_L = 134\,\text{GPa}, \quad E_T = E_Z = 10.2\,\text{GPa}, \quad G_{LT} = G_{LZ} = 5.52\,\text{GPa}, \quad G_{TZ} = 3.43\,\text{GPa},$$

$$\nu_{LT} = \nu_{LZ} = 0.3, \quad \nu_{TZ} = 0.49.$$

To determine the free constants, we match the near-field solution (18) with the remote boundary conditions through boundary collocations. Let c, $\bar{u}$, h_{90}, h_0 and b denote half the crack length, the displacement prescribed at the right end of the unit cell, the thickness of the 90° and 0° ply, and half the spacing of the transverse cracks, respectively (see Fig. 2). The remote boundary conditions for the case of $[90/0]_s$ laminates may then be written as

$$\sigma_{11}(-c, x_2) = \sigma_{12}(-c, x_2) = 0 \quad 0 \leqslant x_2 \leqslant h_{90} \text{ on the left end of the 90° ply;} \tag{19a}$$

$$\sigma_{22}(x_1, h_{90}) = \sigma_{12}(x_1, h_{90}) = 0 \quad -c \leqslant x_1 \leqslant b-c \text{ on the top surface;} \tag{19b}$$

$$u_1(b-c, x_2) = \bar{u}, \quad \frac{\partial u_2(b-c, x_2)}{\partial x_1} = 0 \quad -h_0 \leqslant x_2 \leqslant h_{90} \text{ on the right end;} \tag{19c}$$

$$\frac{\partial u_1(x_1, -h_0)}{\partial x_2} = u_2(x_1, -h_0) = 0 \quad -c \leqslant x_1 \leqslant b-c \text{ on the mid-plane;} \tag{19d}$$

$$u_1(-c, x_2) = \frac{\partial u_2(-c, x_2)}{\partial x_1} = 0 \quad -h_0 \leqslant x_2 \leqslant 0 \text{ on the left end of the 0° ply.} \tag{19e}$$

For $[0/90]_s$ laminates, we have similar boundary conditions. Note that the boundary conditions involve the prescribed displacements for u_1 and u_2, and this requires the inclusion of the rigid body modes in the eigenfunction solution as in eqn (18). We now truncate the series solution (18) to retain a finite number of terms, and determine the free constants γ_{1n}, γ_{2n} and γ_{3n} through the boundary collocations such that the resulting solution may approximate the above remote boundary conditions in the least square sense. The detailed procedure is outlined in Appendix A.

The resulting numerical solutions show that the delamination crack for $[90/0]_s$ is fully opened while it is fully closed for $[0/90]_s$. This is consistent with the following observations: the thickness contraction of the 0° ply due to the axial stress σ_{xx} remains almost uniform along the x_1 coordinate, whilst the thickness contraction of the 90° ply, on the other hand, increases due to shear lag as we proceed away from the damaged zone along the x_1 coordinate. Thus, for $[90/0]_s$ laminates, wherein the 90° ply are free from constraints on the exterior surface, the material elements near the free surface of the 90° ply around the delamination region are pulled toward the undamaged region and this causes the crack to tend to open. For $[0/90]_s$ laminates, wherein the interior 90° ply is symmetrically constrained on the upper and lower faces by the 0° ply, the crack face on the 90° ply is brought into contact with the crack face of the 0° ply because the thickness contraction of the 90° ply due to axial extension is relatively small near the delamination region compared with the undamaged region.

The eigenvalues for the two types of interfacial cracks are given as:

$$n-\tfrac{1}{2}\pm i\eta \text{ and } n \quad \text{for the opened cracks in } [90/0]_s$$

and

$$n-\tfrac{1}{2} \text{ and } n \quad \text{for the closed cracks in } [0/90]_s$$

$$(n = 0, 1, 2, 3, \ldots \quad \text{and} \quad \eta = 0.03\ldots).$$

Notice that the imaginary part of the first eigenvalue is not zero for the opened cracks of $[90/0]_s$ and therefore the stress singularity is oscillatory, while the stress field near the closed cracks of $[0/90]_s$ has the inverse square root singularity. This agrees with the results reported in the earlier literature, e.g. Comninou (1977) and Wang and Choi (1983b). Taking the singular term alone in the eigenfunction expansion (17a,b), we can write the asymptotic stress field ahead of the crack tip and the crack tip opening displacements as

$$\sigma_{\alpha\beta}(r,0) = [A_{\alpha\beta}\cos(\eta\ln r)+B_{\alpha\beta}\sin(\eta\ln r)]r^{-1/2}+\cdots \tag{20a}$$

$$u_\alpha(r,\pi)-u_\alpha(r,-\pi) = [a_\alpha\cos(\eta\ln r)+b_\alpha\sin(\eta\ln r)]r^{1/2}+\cdots$$
$$\text{for the opened cracks in } [90/0]_s \tag{20b}$$

and

$$\sigma_{\alpha\beta}(r,0) = C_{\alpha\beta}r^{-1/2}+\cdots,\quad u_\alpha(r,\pi)-u_\alpha(r,-\pi) = c_\alpha r^{1/2}+\cdots$$
$$\text{for the closed cracks in } [0/90]_s, \tag{20c,d}$$

where "η" is the imaginary part of the stress singularity. The parameters $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta}$, a_α, b_α and c_α, normalized by the applied normal strain $\varepsilon_0 = \bar{u}_0/b$, are tabulated for the case $c/h_{90} = 1$, $b/h_{90} = 4$ and $h_0/h_{90} = 1$ in Table 1. The stress singularity is oscillatory for the opened interfacial cracks and therefore a physically inadmissible interpenetration occurs as r goes to zero. To rectify such a shortcoming, Comninou (1977) proposed a new interfacial crack model for dissimilar isotropic materials, wherein the partially closed crack faces are in frictionless contact near the tips, and Wang and Choi (1983b) extended such a partially closed crack model to the case of dissimilar anisotropic materials. However, the zone of interpenetration is negligibly small in a fully opened crack and the asymptotic stress fields are almost the same for both of the fully opened and the partially closed models (Wang and Choi, 1983b); moreover, the partially closed model also has an inconsistency that it is

Table 1. Asymptotic stress field ahead of the crack tip and crack tip opening displacements
$\varepsilon_0 = \bar{u}/h,\ h_0/h_{90} = 1,\ h_0/h = 0.25,\ c/h = 0.25$

		$[90/0]_s$	$[0/90]_s$
Stress field ahead of the crack tip $\sigma_{r\theta}(r,0)/\varepsilon_0$	A_{11}	0.68946	—
	B_{11}	−0.93051	—
	A_{22}	0.91054	—
	B_{22}	−1.22889	—
	A_{12}	1.44074	—
	B_{12}	1.06751	—
	C_{11}	—	0.
	C_{22}	—	0.
	C_{12}	—	−1.74423
Crack tip opening displacement $[u_r(r,\pi)-u_r(r,-\pi)]/\varepsilon_0$	a_1	0.67692	—
	b_1	0.42412	—
	a_2	−0.79231	—
	b_2	0.58705	—
	c_1	—	−0.80514
	c_2	—	0.

$$\sigma_{r\theta}(r,0)/\varepsilon_0 = [A_{r\theta}\cos(\eta\ln r) + B_{r\theta}\sin(\eta\ln r)]r^{-1/2} + \cdots$$

$$[u_r(r,\pi) - u_r(r,-\pi)]/\varepsilon_0 = [a_r\cos(\eta\ln r) + b_r\sin(\eta\ln r)]r^{1/2} + \cdots$$

for the opened cracks in $[90/0]_s$.

and

$$\sigma_{r\theta}(r,0)/\varepsilon_0 = C_{r\theta}r^{-1/2} + \cdots,\quad [u_r(r,\pi) - u_r(r,-\pi)]/\varepsilon_0 = c_r r^{1/2} + \cdots$$

for the closed cracks in $[0/90]_s$.

not reduced to the solution for a homogeneous body when the two plies become identical. For such reasons, we do not seek the solution for the partially closed model here.

To characterize the stress field near the present delamination cracks, we may use fracture mechanics parameters such as the stress intensity factors K_I, K_{II} and the energy release rates G. For closed cracks in $[0/90]_s$, there are no difficulties introducing the classical stress intensity factors for this purpose. In fully opened interface cracks in $[90/0]_s$, however, modes I and II are inherently coupled, and the stress intensity factor is not defined in the same way as in the classical fracture mechanics. One possible definition, as suggested by Rice (1988), is to define the stress intensity factor by obtaining the stress value at a specific small distance from the crack tip, that is, for a very small value of $\hat{r}$,

$$K_I = \sqrt{2\pi\hat{r}}\,\sigma_{22}(\hat{r},0),\quad K_{II} = \sqrt{2\pi\hat{r}}\,\sigma_{12}(\hat{r},0).$$

For the present case, we take $\hat{r}$ to be $c/50$. Because the imaginary part of singularity "η" is very small ($\eta = 0.03\ldots$), changing the value of $\hat{r}$ within a modest range, say, an order of a factor of 10 or less, does not affect the values of K_I and K_{II} significantly (Rice, 1988).

To confirm the convergence of the solution, we computed the stress intensity factors and the maximum mismatch of the remote boundary conditions, varying the number of the eigenvalues and the number of the collocation stations (Table 2). We see that the present solution has a very stable convergence characteristic. The accuracy of the solution has been confirmed through a comparison with the result obtained from the singular hybrid element technique (Jang, 1990): the stress intensity factors K_I and K_{II} from the two numerical solutions agree up to the first two digits and the relative difference is less than two per cent, when the boundary collocation solution with the 63 terms truncated in the eigenfunction series is compared with the solution from the F.E.M. model comprising 52 regular elements and one singular element. Such an excellent agreement between the solutions from the different approaches has been reported for a stress field near cracks normal to the ply interface (Im and Kim, 1989) and for a stress field near the free edge (Stolarski and Chiang, 1989).

As discussed earlier, the rigid translational modes, which apparently meet the near-field conditions (12) through (14a,b), are not included in eqn (17a) due to the inherent

Table 2. Solution convergence versus the number of eigenvalues and the number of collocation stations

$\varepsilon_0 = \bar{u}/b,\ c/h_{90} = 1,\ b/h_{90} = 4,\ h_0/h_{90} = 1$

$[90/0]_s$ unit: GPa $(\text{mm})^{1/2}$

No. of eigenvalues	No. of collocation stations	K_I/ε_0	K_{II}/ε_0	Max. mismatch
51	100	2.6514	3.0656	2.11%
51	120	2.6514	3.0660	1.58%
51	140	2.6512	3.0662	1.57%
63	100	2.7016	3.0511	1.50%
63	120	2.6862	3.0844	1.07%
63	140	2.6606	3.1140	1.27%
110	100	2.7207	3.0291	1.16%
110	120	2.6652	3.0598	1.15%
110	140	2.6867	3.0708	1.14%

$$\text{Eigenvalues:}\quad \delta = \begin{cases} n & (n = 0, 1, 2, 3, \ldots \\ (n-\frac{1}{2}) \pm i\eta & \eta = 0.03 \ldots). \end{cases}$$

$[0/90]_s$

No. of eigenvalues	No. of collocation stations	K_{II}/ε_0	Max. mismatch
51	100	−1.8063	1.70%
51	120	−1.7733	1.06%
51	140	−1.7751	1.02%
63	100	−1.8105	1.26%
63	120	−1.7682	1.47%
63	140	−1.8090	1.03%
110	100	−1.7706	0.53%
110	120	−1.7876	1.10%
110	140	−1.8070	1.21%

$$\text{Eigenvalues:}\quad \delta = \begin{cases} n \\ n-\frac{1}{2} \end{cases} \quad (n = 0, 1, 2, 3, \ldots).$$

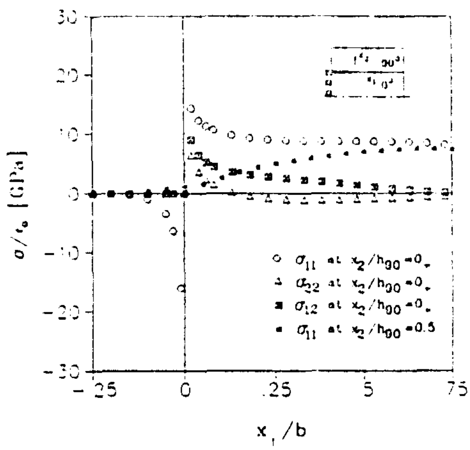

Fig. 3. Stress distribution for $[90/0]_s$.

limitation of the functional from (10) or (11a). Thus we can obtain an accurate solution in the light of completeness only when we add these terms to the expression for displacements, as in eqn (18). Indeed, numerical results show that the maximum mismatch of the remote boundary conditions increases by a factor of 10 and the stress intensity factors deviate more than 30% when the rigid body displacements are omitted, although the solution shows a good convergent behavior depending upon the numbers of eigenvalues and collocation stations.

Figures 3 through 6 show the stress distribution along the x_1 axis at $x_2/h_{90} = 0$ (the

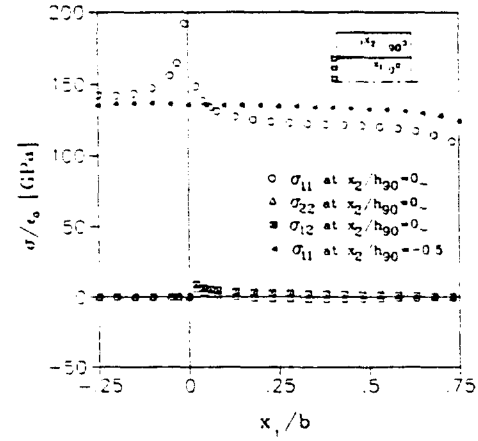

Fig. 4. Stress distribution for $[90/0]_s$.

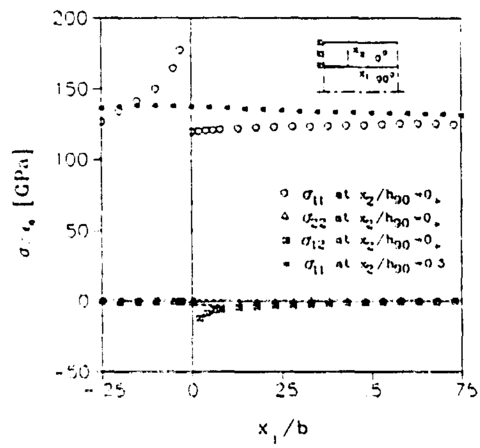

Fig. 5. Stress distribution for $[0/90]_s$.

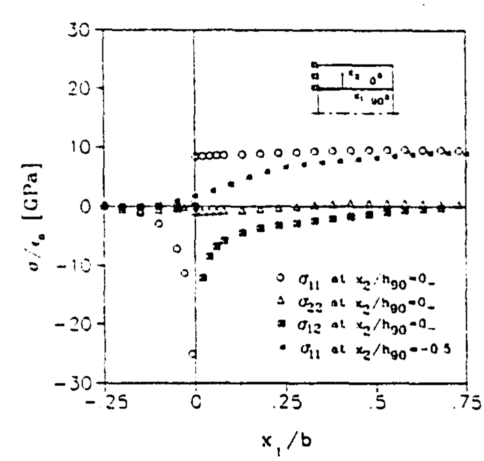

Fig. 6. Stress distribution for $[0/90]_s$.

ply interface) and at $x_2/h_{90} = \pm 0.5$. For the closed cracks in $[0/90]_s$, the normal stress is compressive, and the shear mode is responsible for delamination growth. It is also noticed for the opened cracks in $[90/0]_s$ that the interlaminar shear stress component σ_{12} is greater than the normal stress component σ_{22}, and this suggests that the high shear stress is also an important driving force for the growth of opened delamination cracks. The characteristics of the stress distributions σ_{22} and σ_{12}, near the closed crack tip of $[0/90]_s$, agree with the

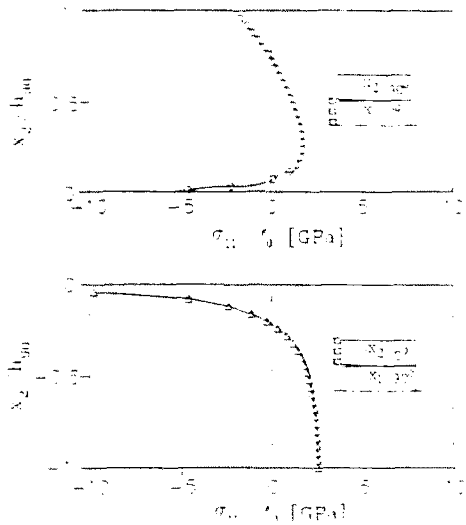

Fig. 7. Stress distribution through the thickness at the crack tip.

results for a partially closed interface crack in isotropic bi-materials (Comninou, 1978), in that σ_{12} is singular at $x_1 = 0_+$, $x_2 = 0$, while σ_{22} is singular at $x_1 = 0_-$ and $x_2 = 0$ and bounded at $x_1 = 0_+$ and $x_2 = 0$. It is noticed that both $[90/0]_s$ and $[0/90]_s$ have peculiar axial stress distributions on the crack face of the 90° ply due to the complexity of loading and geometry. For $[90/0]_s$, the stress component σ_{11} at $x_2 = 0_+$ goes to negative infinity first as the crack tip is approached from the negative x_1 axis. It then undergoes a violent oscillation until it ends up at positive infinity as we approach the crack tip from the positive x_1 axis. For $[0/90]_s$, the stress component σ_{11} at $x_2 = 0_-$ goes to negative infinity with no oscillations this time as the crack tip is approached from the negative x_1 axis but it is bounded as we approach the crack tip from the positive x_1 axis. The large negative values of the axial stress on the crack face of the 90° ply as the crack tip is approached from the negative x_1 axis appear to be contrary to our intuition. However, such negative axial stress may be explained when we examine the stress distribution through the thickness of the 90° ply at the crack tip ($x_1 = 0$) and the global force equilibrium of the part of the 90° ply comprising $-c \leqslant x_1 \leqslant 0$. Because there is no axial force through the thickness on the left end ($x_1 = -c$) of the part, the net axial force through the thickness on the right end ($x_1 = 0$) must be zero. However, the axial stress on the 90° ply, which continuously decreases as the crack tip is approached from the right end ($x_1 = b-c$) does not disappear exactly at the cross-section of $x_1 = 0$ where the crack tip is located. That is, the axial stress σ_{11} does not decrease to zero but still remains positive on the greater part of the cross-section at $x_1 = 0$. To maintain the overall force equilibrium of the part considered above in the x_1 direction, there must be a portion of cross-section under large compressive axial stress (see Fig. 7). For the $[90/0]_s$ laminates, the stress distribution through the thickness should be such that it alone may meet the overall moment equilibrium in addition to the force equilibrium, and hence it is more complicated than the case of $[0/90]_s$ wherein the moment equilibrium is automatically satisfied due to symmetry with respect to the mid-plane.

For both opened and closed delamination cracks, the energy release rates G can be calculated from Irwin's virtual crack extension concept. For an opened delamination crack, modes I and II are strongly coupled and the calculation of the energy release rates involves a substantial amount of algebra including computation of a complex contour integral (Willis, 1971). For self-sufficiency of development, the expressions for the energy release rates are outlined in Appendix B. Figures 8 and 9 show the stability of delamination cracks in terms of energy release rates under a fixed load condition. These figures reveal important features regarding the growth of delamination cracks originating from transverse cracking in cross-ply laminates. When we assume a failure criterion based upon the critical energy release rate, $G = G_c$ (G_c is a material constant), we see that any delamination crack will

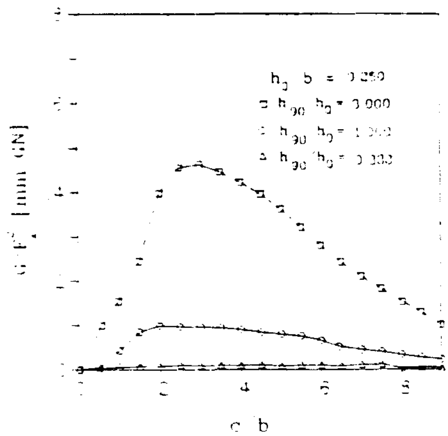

Fig. 8. Energy release rate versus the length of delamination crack under a fixed loading condition in $[90/0]_s$.

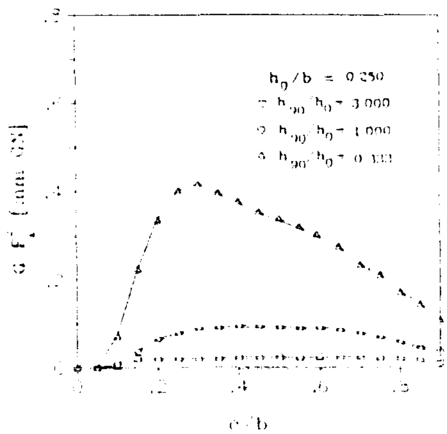

Fig. 9. Energy release rate versus the length of delamination crack under a fixed loading condition in $[0/90]_s$.

undergo unstable growth until the crack length reaches a critical value associated with the maximum G. The crack will continue to grow under a fixed loading until the energy release rate decreases to the critical value G_c. Thus, there exists an inherently built-in crack arrest mechanism for the present delamination cracks. The critical crack length is dependent on the relative crack spacing b/h_{90} as well as on the relative ply thickness h_0/h_{90}. As the thickness of the 90° ply increases relative to the 0° ply, the energy release rate notably increases and therefore the crack growth after the onset of the delamination crack will be more unstable. This is consistent with the observation that a larger portion of the load will be carried by the 90° layer as the thickness of the 90° ply increases, and the energy release rate will accordingly increase under a fixed load condition. On the other hand, it is worthwhile to note that the energy release rate tends to become independent of the crack length, except when the crack is of the order of the thickness of the 90° ply, as the thickness of the 90° ply decreases.

The energy release rates are zero at the onset of the delamination cracks, as shown in Figs 8 and 9, as the stress singularity of the transverse cracks in a cross-ply laminate under extension is weaker than the inverse square root singularity and therefore the stress intensity for a vanishing delamination crack approaches zero under extension (Im and Kim, 1989).

Acknowledgements—The authors gratefully acknowledge the financial support for this work, provided by the Korea Science and Engineering Foundation (KOSEF). They thank professor T. C .T. Ting of The University of Illinois at Chicago for the fruitful discussion.

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APPENDIX A

The free constants γ_{kn} $(k = 1, 2, 3)$ in eqn (16) are determined through boundary collocation so that the eigenfunction solution (17b) and (18) meet the remote boundary conditions (19a–e) in the least square sense. For convenience, we write eqns (17b) and (18) as

$$\sigma_i = \sum_{n=1}^{t} \beta_n g_i^n(x_1, x_2, \delta_n), \quad u_j = \sum_{n=1}^{t} \beta_n h_j^n(x_1, x_2, \delta_n) \tag{A1}$$

where β_n denote one of the γ_{kn} or u_j^0, and g_i^n, h_j^n represent the corresponding eigenfunctions in the expressions for $Q_{ns}^{(m)}$ and $P_{n\alpha\beta}^{(m)}$. Let $\bar{\sigma}_i$, $\bar{u}_j$, $\partial\bar{u}_j/\partial x_1$ and $\partial\bar{u}_j/\partial x_2$ $(i = 1$–$6,\ j = 1$–$3)$ denote the prescribed values for the stresses, displacements and displacement derivatives on the remote boundaries. The boundary conditions (19a–e) can then be written in the form

$$\bar{\sigma}_i = \sum_{n=1}^{t} \beta_n g_i^n(\bar{x}_1, \bar{x}_2, \delta_n), \quad \bar{u}_j = \sum_{n=1}^{t} \beta_n h_j^n(\bar{x}_1, \bar{x}_2, \delta_n)$$

$$\frac{\partial \bar{u}_j}{\partial x_1} = \sum_{n=1}^{t} \beta_n h_{j,1}^n(\bar{x}_1, \bar{x}_2, \delta_n), \quad \frac{\partial \bar{u}_j}{\partial x_2} = \sum_{n=1}^{t} \beta_n h_{j,2}^n(\bar{x}_1, \bar{x}_2, \delta_n)$$

where $\bar{x}_1$, $\bar{x}_2$ denote the dimensionless coordinates of the remote boundaries, and $h_{j,1}^n$, $h_{j,2}^n$ are the partial derivatives of h_j^n with respect to the x_1, x_2 coordinates, respectively.

The functional to be minimized in the boundary collocation is written as

$$\Pi = \int_{\partial B_*}\left[\sum_{i=1}^{6} A_i\left(\bar{\sigma}_i - \sum_{n=1}^{\ell} \beta_n g_i^n\right)^2 + \sum_{i=1}^{3} B_i\left(\bar{u}_i - \sum_{n=1}^{\ell} \beta_n h_i^n\right)^2 + \sum_{i=1}^{3} C_i\left(\frac{\partial \bar{u}_i}{\partial x_1} - \sum_{n=1}^{\ell} \beta_n h_{i,1}^n\right)^2 + \sum_{i=1}^{3} D_i\left(\frac{\partial \bar{u}_i}{\partial x_2} - \sum_{n=1}^{\ell} \beta_n h_{i,2}^n\right)^2\right] \mathrm{d}s. \quad \text{(A2)}$$

Here ∂B_* denotes the whole remote boundaries, and A_i, B_i, C_i and D_i are equal to 1 if the terms correspond to the prescribed boundary conditions, and otherwise become zero. For example, eqns (19a–e) indicate, for the [90/0]$_s$ laminates, that $A_1 = A_6 = 1$ and the other terms are zero on the left end of the 90° ply, $A_2 = A_6 = 1$ and the others are zero on the top surface of the laminates, $B_1 = C_2 = 1$ and the others are zero on the right end, etc. Minimization of the above residual leads to the following system of simultaneous linear equations for the free constants β_n:

$$\sum_{n=1}^{\ell} \beta_n \int_{\partial B_*} \sum_{i=1}^{3} [A_i g_i^m g_i^n + A_{i+3} g_{i+3}^m g_{i+3}^n + B_i h_i^m h_i^n + C_i h_{i,1}^m h_{i,1}^n + D_i h_{i,2}^m h_{i,2}^n]\, \mathrm{d}s$$
$$= \int_{\partial B_*} \sum_{i=1}^{3}\left[A_i \bar{\sigma}_i g_i^m + A_{i+3}\bar{\sigma}_{i+3} g_{i+3}^m + B_i \bar{u}_i h_i^m + C_i \frac{\partial \bar{u}_i}{\partial x_1} h_{i,1}^m + D_i \frac{\partial \bar{u}_i}{\partial x_2} h_{i,2}^m\right] \mathrm{d}s \quad (m = 1, 2, 3, 4 \ldots). \quad \text{(A3)}$$

The system of the above infinite number of linear equations can be approximated by an appropriate truncation, say $m, n = 1$–N, and in general, a proper scaling procedure is required for successful numerical computation.

APPENDIX B

According to Irwin's virtual crack extension concept, the energy release rate for the case of plane strain deformation may be written as

$$G = G_{\mathrm{I}} + G_{\mathrm{II}} = \lim_{\delta r \to 0} \frac{1}{2\delta r} \int_0^{\delta r} [\sigma_{22}(r, 0)\{u_2(\delta r - r, \pi) - u_2(\delta r - r, -\pi)\} + \sigma_{12}(r, 0)\{u_1(\delta r - r, \pi) - u_1(\delta r - r, -\pi)\}]\, \mathrm{d}r. \quad \text{(A4)}$$

Substituting eqns (20a–d) into this expression, and performing lengthy but straightforward algebra, we obtain the form:

$$G = \lim_{\delta r \to 0} \frac{1}{2\delta r} \int_0^{\delta r} \left[m_1 \cos\left\{\eta \ln\left(\frac{\delta r - r}{r}\right)\right\} + m_2 \sin\left\{\eta \ln\left(\frac{\delta r - r}{r}\right)\right\}\right] \sqrt{\frac{\delta r - r}{r}}\, \mathrm{d}r \quad \text{for an opened crack} \quad \text{(A5)}$$

and

$$G = \lim_{\delta r \to 0} \frac{1}{2\delta r} \int_0^{\delta r} m_3 \sqrt{\frac{\delta r - r}{r}}\, \mathrm{d}r = \frac{1}{4}\pi m_3 \quad \text{for a closed crack,} \quad \text{(A6)}$$

where

$$m_1 = \tfrac{1}{2}(A_{22}a_2 + A_{12}a_1 + B_{22}b_2 + B_{12}b_1)$$
$$m_2 = \tfrac{1}{2}(A_{22}b_2 + A_{12}b_1 - B_{22}a_2 - B_{12}a_1)$$
$$m_3 = C_{22}c_2 + C_{12}c_1$$

and $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta}$, a_α, b_α and c_α are the coefficients appearing in eqns (20a–d). With the aid of contour integration, Willis (1971) showed that the integral (A5) leads to

$$G = \tfrac{1}{4}\pi \frac{m_1 + 2m_2\eta}{\cosh(\eta\pi)},$$

which is reduced to (A6) when the oscillation disappears ($\eta = 0$).